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Internal avalanches in a granular medium

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Avalanches of grain displacements can be generated by creating local voids within the interior of a granular material at rest in a bin. Modeling such a two-dimensional granular system by a collection of monodisperse discs, the system, on repeated perturbations, shows all signatures of self-organized criticality. During the propagation of avalanches the competition among grains creates arches and in the critical state a distribution of arches of different sizes is obtained. Using a cellular automata model we demonstrate that the existence of arches determines the universal behavior of the model system. [S1063-651X(98)51212-3]

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The search for ''self-organized criticality'' (SOC) [1] in granular systems in general, and in sandpiles in particular, has been a subject of active research over the last decade. How such a system reacts in the form of fast cascades of grain displacements, called ''avalanches,'' in response to a slow external drive at the microscopic level, is the crucial question of study. It has been suggested that, starting from an arbitrary initial condition, a nonequilibrium critical state, characterized by scale-free avalanches in both space and time, should be expected after a long time [1,2]. Other naturally occuring physical phenomena, such as forest fires [3], river networks [4], earthquakes [5], etc., have also been proposed as examples of systems showing SOC.

It was observed that a steadily shaped sandpile, grown on a fixed base, fulfills all of the requirements of SOC. In this state, dropping even a few grains creates rapidly moving avalanches of sand sliding along the surface of the pile. It was expected that the avalanches should be power law distributed in their spatial, as well as temporal, extents and, therefore, a sandpile should be considered a simple example of SOC [1,2].

Experimental observations, however, show partial support of this idea. Sand is allowed to fall from a slowly rotating semicylindrical drum through the space between the plates of a vertical parallel plate capacitor. The Fourier spectrum of the time series data of fluctuating capacitance showed a peak contrary to the expectation of a power law distribution [6]. Similarly, sand dropping from the edge of a sandpile on a fixed base was directly measured, and was seen to have a power law distribution only for small systems [7]. It was argued that, due to the approximately spherical shape of the grains used in these experiments, the effect of inertia cannot be neglected, and this was the reason for the absence of scaling behavior. This is verified in an experiment using rice grains, which are highly anisotropic, and criticality was observed [8].

A number of theoretical models, generally known as ''sandpile'' models, have been studied. The models are based on stochastically driven cellular automata, which evolve under a nonlinear, diffusive mechanism, leading to a nonequilibrium critical state. The most widely studied is the Abelian sandpile model (ASM), in which the stable configuration does not depend on the sequence of sand grain additions [9]. Other variants of the sandpile model include situations in which the stability of a sand column depends on the local slope or the Laplacian of the height profile [10]. A two-state sandpile model with stochastic evolution rules is also studied [11].

In all of the studies discussed above, the avalanches propagate on the surfaces of the sandpiles. However, there exists the possibility of creating avalanches in the interior of a granular material. In a granular material kept in a bin at rest, different grains support one another by mutually acting balanced forces. If a grain is removed, the grains that were supported by it become unstable and tend to move. Eventually the grains in the farther neighborhood also lose their stability. As a result, an avalanche of grain displacements takes place, which stops when no more grains remain unstable. The basic physical behavior here is thus quite different from the avalanches on the surface of the pile because of the constraints to particle motion in the dense particle beds.

A two-dimensional semilattice model was studied for this problem [12]. Nonoverlapping unit square boxes model the grains, whose horizontal coordinates can vary continuously, whereas the vertical coordinates are discretized. A grain can only fall vertically if insufficiently supported, and sufficient space below is available. The system is disturbed by removing grains at the bottom and thus creating avalanches of grain movements.

During the propagation of avalanches inside a granular medium, grains compete locally with one another to occupy the same vacant space. The high packing densities of the particle beds prevent a single particle from occupying the available void space and, consequently, particles get locked to form ''arches'' [13]. A stable arch is a chain of grains in which the weight of each grain is balanced by the reaction forces from two neighboring grains in the chain. Arches can form only when a grain is allowed to roll over other grains. Since the rolling motion of the grains was absent, the arches were not formed in the granular patterns [12].

Here we study a more realistic model of this problem in two dimensions, where both fall and roll motions of grains

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are allowed. The granular system is represented by N hard monodisperse discs of radii R. No two grains are allowed to overlap, but they can touch each other and one can roll over the other without slipping if possible. A rectangular area of size $L\times L$ on the x-y plane represents our two-dimensional bin. The periodic boundary condition is used along the x direction, and the gravity acts along the -y direction. The bottom of the bin at $y=0$ is highly sticky and any grain that comes into contact with it gets stuck there and cannot move any further.

The dynamical evolution of the system is studied by a ‘‘pseudodynamics’’ [14]. Unlike the method of molecular dynamics, we do not solve here the classical equations of motion for the grain system. Only the direction of gravity and the local geometrical constraints, due to the presence of other grains, govern the movement of a grain. To justify using the pseudodynamics we argue that due to the high compactness of the system a grain never has a sufficient amount of time to accelerate much. Therefore, in our model, a grain can have only two types of movements in unit time: it can either *fall* vertically up to a maximum distance δ, or it can *roll* up to a maximum angle $\theta=\delta/2R$ over another grain in contact. However, for the whole system, some disks can fall, some others can roll, and the rest remain stable in unit time.

Movement of a grain requires information on the positions of other grains in the neighborhood that it may possibly interact with. An efficient way to search this is to digitize the bin into a square grid and associate the serial number of a grain to the primitive cell of the grid containing its center. The choice of $R=1/\sqrt{2}+$ ensures that a cell corresponds to, at most, one grain. With this choice it is sufficient to search only within the 24 neighboring cells for possible contact grains. The weight of a grain n is supported by two other grains. If n_L and n_R are the serial numbers of the left (L) and right (R) supporting grains, then the grain n is updated as follows: (i) if $n_L=n_R=0$, it falls; (ii) if $n_L\neq 0$, but $n_R=0$, it rolls over n_L; (iii) if $n_L=0$, but $n_R\neq 0$, it rolls over n_R; (iv) if $n_L\neq 0$ and $n_R\neq 0$ it is stable.

When the grain n, with the center at (x_n,y_n), is allowed to fall, it may come in contact with a grain r at (x_r,y_r) within the distance δ. The new coordinates are

$$y_n'=y_r+\sqrt{4R^2-(x_n-x_r)^2} \quad \text{if} \quad y_n-y_n'<\delta$$

$$\text{otherwise,} \quad y_n'=y_n-\delta.$$

Similarly, the grain n, while rolling over the grain r, may come in contact with another grain t at (x_t,y_t). The new coordinates for the center of n at which it touches both r and t are

$$x_n'=\frac{1}{2}(x_t+x_r)+g(y_t-y_r)\sqrt{\frac{4R^2}{d_{rt}^2}-\frac{1}{4}},$$

$$y_n'=\frac{1}{2}(y_t+y_r)-g(x_t-x_r)\sqrt{\frac{4R^2}{d_{rt}^2}-\frac{1}{4}}.$$

Here d_{rt} is the distance between the centers of the grains r and t, and $g=+1$ and -1 for left and right rolls. To reach the new position, if the grain n rolls an angle $\theta_m<\theta$, it is accepted; otherwise it rolls up to an angle θ.

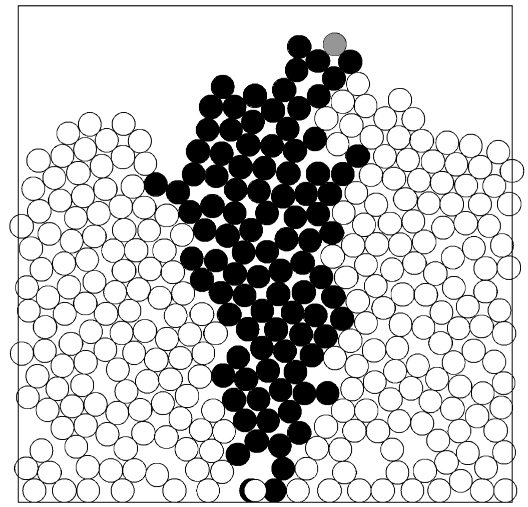

FIG. 1. Picture of a typical avalanche. Open circles denote the undisturbed grains; closed circles denote the displaced grains; opaque circle at the bottom denotes the position of the grain that was removed; shaded circle denotes the position of the grain at which the removed grain was replaced. In a system of size $L=30$, with the number of grains $N=340$, 97 grains took part in the avalanche.

The initial grain pattern is generated by the ballistic deposition method with restructuring [15,16]. Grains are released sequentially one after another along randomly positioned vertical trajectories, and are allowed to fall until they touch the growing pile, and then roll down along the paths of steepest descent to their local stable positions. We notice that in the initial pattern no big arch exists, since a grain, while rolling down along the surface, does not need to compete with any other grain. Using a system of $L=80$, we compute the packing fraction $\rho=0.822\pm 0.005$, which is consistent with the more precise estimate of 0.8180 ± 0.0002 in [16].

The system is repeatedly perturbed by removing grains randomly at the bottom, one after the other. Every time a grain is removed, an avalanche follows, and after the avalanche is over the removed grain is placed back at a random position on the top surface again, using the same ballistic method (Fig. 1). We first observe that ρ of the stable configuration, averaged over many initial random patterns generated with the same algorithm, decreases with the number of avalanches created, and finally reaches a steady value of 0.748 ± 0.005. A similar study but with different initial configurations with a different value of average initial ρ shows that the final steady state packing fraction reaches the same value, which implies that the final stable state is independent of the initial state. The total number of grain displacements is called the avalanche size s, and the duration of the avalanche is the lifetime t. Power law distributions are observed for s and t: $D(s)\sim s^{-\tau_s}$ and $D(t)\sim t^{-\tau_t}$, with $\tau_s\approx 1.7$ (Fig. 2) and $\tau_t\approx 2.1$. The peak at the right end of the distribution is due to the finite size of the system. Upon increasing the system size, the position of the peak shifts to the right and the scaling region becomes longer.

The observed exponents in this model are significantly larger than the corresponding values of $\tau_s\approx 1.34$ and τ_L

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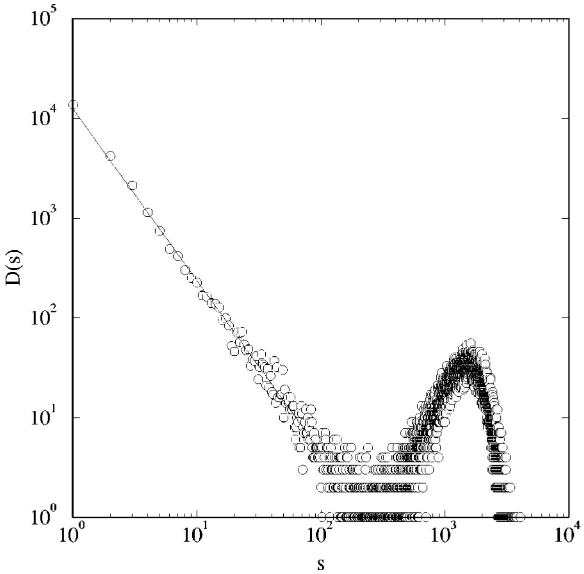

FIG. 2. Plot of the avalanche size distribution $D(s)$ vs s for a system of size N=10 000, and over 70 000 avalanches. The straight part fits with a slope $\tau_s = 1.7$.

$\approx$1.47 in [12]. We argue that the reason for this difference could be the existence of arches in our model. Due to the arches, the propagation of avalanches is arrested more frequently than in [12] and, therefore, smaller avalanches are more probable, which enhances the values of the critical exponents in our model.

To demonstrate more explicitly that the above reasoning may be correct, we study a cellular automata model for the granular systems. A square lattice of size L, with periodic boundary condition along the x axis, represents the bin. The gravity acts in the -y direction. Positions of the grains are limited only to the lattice sites: a site can either be occupied by a grain or remain vacant. Initially, the bin is filled up again using the ballistic method.

A single movement of a grain at $C(i,j)$ involves the neighboring seven sites: $LU(i-1,j+1), L(i-1,j), LD(i-1,j-1), D(i,j-1), RD(i+1,j-1), R(i+1,j)$, and $RU(i+1,j+1)$. In the *fall* move the grain comes down one level

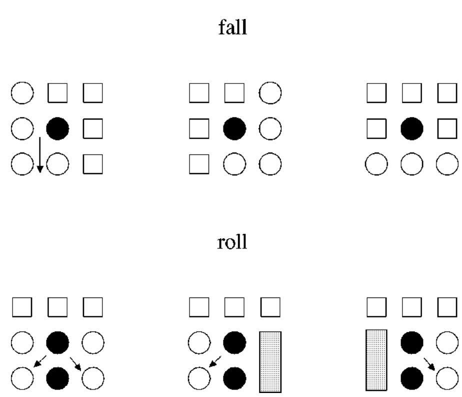

FIG. 3. Possible *fall* and *roll* moves in the cellular automata model of the granular system on the square lattice. A filled circle denotes the position of a grain, an unfilled circle denotes a vacant site. The grain moves to the vacant position irrespective of the occupation of the sites with square boxes. A shaded rectangle denotes a pair of sites, in which at least one is occupied.

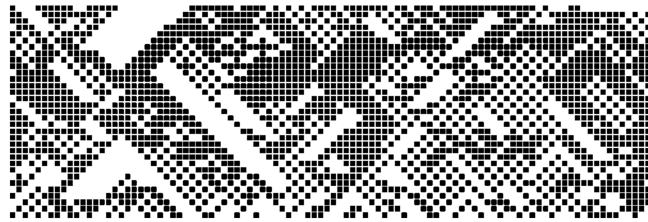

FIG. 4. Stable configuration of the granular system in the cellular automata model A after a large number of avalanches are created. Triangular arches are noticed.

to the vacant site at D and in the *roll* move the grain goes to the vacant site either at LD or at RD (Fig. 3). An arch is formed when a grain at C is considered stable if both of its two diagonally opposite sites either at LD and RU or at LU and RD, are occupied. As a result, on the square lattice, the only possible shape of an arch consists of two sides of a triangle. Depending on whether or not we allow the arch formations, we define the following two models.

In model A we allow arch formations. The grain at C falls only if any of the following three conditions is satisfied: (i) LD, L, and LU are vacant, (ii) RD, R and RU are vacant (iii) LD and RD are vacant. In all other situations the grain does not fall. Notice that in conditions (i) and (ii) we are allowing the formation of arches. The grain at C rolls only if site D is occupied. This is done in any of three ways: (i) if both LD and L are vacant, but either of RD and R is occupied, then the grain rolls to LD; (ii) if both RD and R are vacant, but either of LD and L is occupied, then the grain rolls to RD; (iii) if all four sites at LD, L, RD and R are vacant, then the grain rolls either to LD or to RD with a probability of 1/2. A steady state pattern is shown in Fig. 4.

In model B we do not allow arch formations. The first two conditions for fall of model A are modified as (i) LD and L are vacant, (ii) RD and R are vacant. All other conditions of fall, as well as roll, remain the same as in model A.

Initial granular patterns are generated using the same random ballistic deposition method with restructuring [15,16], and patterns are the same for both models. As before, the avalanches are created by taking out one grain at a time at the bottom, allowing the system to relax, and replacing it randomly on the surface after the avalanche is over. Here we also see that the average density of sites, starting from an initial value of 0.907$\pm$0.005, decreases to the final stable value of 0.590$\pm$0.005 in model A, and to 0.618$\pm$0.005 in model B. The avalanche size s and lifetime t follow power law distributions: $D(s) \sim s^{-\tau_s^A}, D(t) \sim t^{-\tau_t^A}$, and similarly for model B. Different exponents are obtained for the two models: $\tau_s^A \approx 1.48$ and $\tau_t^A \approx 1.99$ whereas $\tau_s^B \approx 1.34$ and $\tau_t^B \approx 1.50$. We explain that absence of arches makes the exponent values for the model B close to that of [12], but their presence enhances the values in model A.

To summarize, avalanches of grain displacements can be created in the interior of a granular material at rest by locally disturbing the system. In a numerical study we provide indications that on repeated creations of such avalanches, the granular system reaches a critical state characterized by longrange correlations. The presence of arches plays an important role in determining the critical behavior. Since in an avalanche grain motions are highly constrained, the effect of inertia may not be very significant. Therefore, we conjecture that in real experiments of internal avalanches it should be possible to observe SOC even with spherical grains, unlike the case of surface avalanches, in which anisotropic grains are necessary to observe criticality.

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